Distance to the Decision Boundary

Suppose we have the following classification model:

$$y_i = \begin{cases} +1 & if\ x_i{}^T\beta > 0 \\ -1 & otherwise \end{cases}$$

This note illustrates why the distance from x_i to the decision boundary is given by $\frac{|x_i{}^T\beta|}{\|\beta\|}$ where $\|\beta\| = \sqrt{\beta^T\beta}$.

Suppose we're working in two dimensions and we want to know the distance of $x_i = \begin{bmatrix}1\\3\end{bmatrix}$ to the decision boundary defined by $\beta = \begin{bmatrix}1\\1\end{bmatrix}$. In the graph below, β is shown as the solid black arrow (pointing in the direction of the positive class) and x_i is shown as the blue arrow. The decision boundary is defined by values of x_i where $x_i{}^T\beta$ == 0, so we're looking for vectors along the line $x_{i,2} = -x_{i,1}$; e.g. $s = \begin{bmatrix}-2\\2\end{bmatrix}$ (shown as the dotted black arrow in the graph).

The distance from x_i to the decision boundary is given by the distance of x_i to the orthogonal projection of x_i onto s:

$$v = \frac{x_i{}^T s}{\|s\|}\frac{s}{\|s\|}$$

v is shown as the solid red arrow in the graph below. The cosine of the angle between v and x_i is $\frac{\|v\|}{\|x_i\|} = \frac{x_i{}^T s}{\|x_i\|\,\|s\|}$, so v is simply a scaled version of $\frac{s}{\|s\|}$.

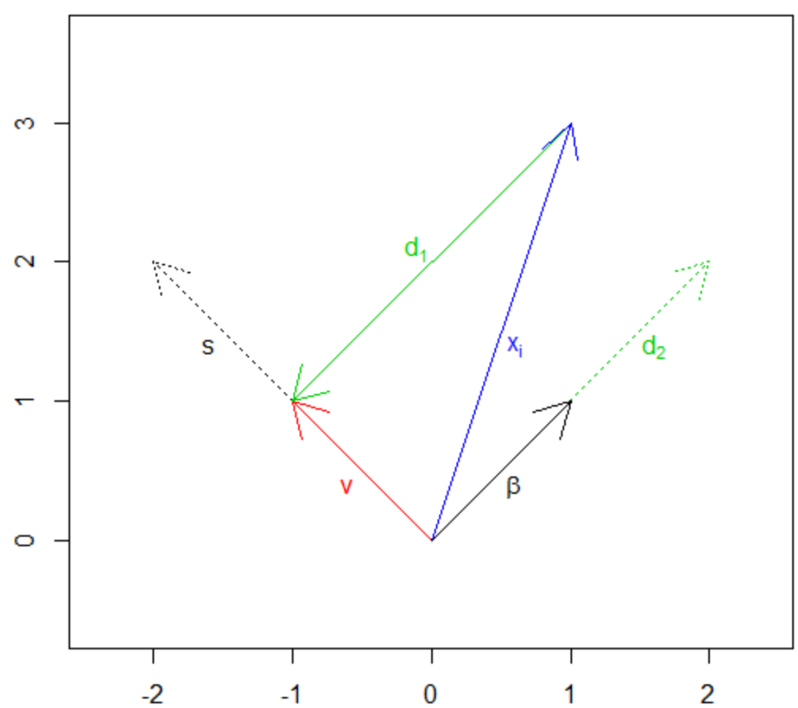
The length of $d_1 = x_i - v$ is the distance of x_i to the decision boundary. This is shown by the solid green arrow in the graph below.

By symmetry, the length of d_1 is the same as the length of the projection of x_i onto β: $d_2 = \frac{x_i{}^T\beta}{\|\beta\|}\frac{\beta}{\|\beta\|}$

d_2 is shown as the dotted green arrow in the graph below.

Since d_2 is simply a scaled version of a unit length version of β, the length of d_2 is given by $\|d_2\| = \frac{|x^T\beta|}{\|\beta\|}$

Notes: $\beta^T\beta$ == $\|\beta\|^2$; distance must be non-negative; and we get the same v if we choose $s = \begin{bmatrix}1\\-1\end{bmatrix}$.



```r
# R code for the graph
windows(6, 6)
library(MASS)
beta = c(1, 1)
s = c(-2, 2)
x = c(1, 3)
v = ((t(x) %*% s) / (t(s) %*% s)) * s
d1 = x - v
d2 = ((t(x) %*% beta) / (t(beta) %*% beta)) * beta
eqscplot(c(-2.5, 2.5), c(-0.5, 3.5), typ = "n")
arrows(0, 0, s[1], s[2], col = "black", lty = "dotted")
arrows(0, 0, v[1], v[2], col = "red")
arrows(x[1], x[2], v[1], v[2], col = "green3")
arrows(0, 0, d2[1], d2[2], col = "green3", lty = "dotted")
arrows(0, 0, x[1], x[2], col = "blue")
arrows(0, 0, beta[1], beta[2], col = "black")
text(-1.6, 1.4, "s", col = "black")
text(-0.6, 0.4, "v", col = "red")
text(0.6, 1.4, expression(x[i]), col = "blue")
text(0.6, 0.4, expression(beta), col = "black")
text(-0.1, 2.1, expression(d[1]), col = "green3")
text(1.6, 1.4, expression(d[2]), col = "green3")
```